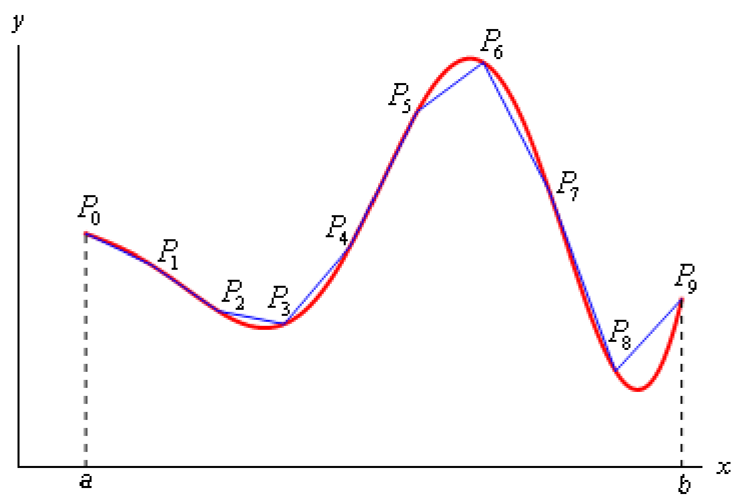
Section 2-1 : Arc Length

In this section we are going to look at computing the arc length of a function. Because it's easy enough to derive the formulas that we'll use in this section we will derive one of them and leave the other to you to derive.

We want to determine the length of the continuous function $y = f(x)$ on the interval $[a, b]$. We'll also need to assume that the derivative is continuous on $[a, b]$.

Initially we'll need to estimate the length of the curve. We'll do this by dividing the interval up into n equal subintervals each of width Δx and we'll denote the point on the curve at each point by P_i. We can then approximate the curve by a series of straight lines connecting the points. Here is a sketch of this situation for $n = 9$.



Now denote the length of each of these line segments by $|P_{i-1}\ P_i|$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

and we can get the exact length by taking n larger and larger. In other words, the exact length will be,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$. We can then compute directly the length of the line segments as follows.

$$|P_{i-1}\ P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

By the **Mean Value Theorem** we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that,

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\Delta x$$

Therefore, the length can now be written as,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + [f'(x_i^*)]^2 \Delta x^2} \\ &= \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x \end{aligned}$$

The exact length of the curve is then,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\ \Delta x \end{aligned}$$

However, using the **definition of the definite integral**, this is nothing more than,

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

A slightly more convenient notation (in our opinion anyway) is the following.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

In a similar fashion we can also derive a formula for $x = h(y)$ on $[c, d]$. This formula is,

$$L = \int_c^d \sqrt{1 + [h'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

Again, the second form is probably a little more convenient.

Note the difference in the derivative under the square root! Don’t get too confused. With one we differentiate with respect to x and with the other we differentiate with respect to y. One way to keep the two straight is to notice that the differential in the “denominator” of the derivative will match up with the differential in the integral. This is one of the reasons why the second form is a little more convenient.

Before we work any examples we need to make a small change in notation. Instead of having two formulas for the arc length of a function we are going to reduce it, in part, to a single formula.

From this point on we are going to use the following formula for the length of the curve.

Arc Length Formula(s)

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

Note that no limits were put on the integral as the limits will depend upon the ds that we're using. Using the first ds will require x limits of integration and using the second ds will require y limits of integration.

Thinking of the arc length formula as a single integral with different ways to define ds will be convenient when we run across arc lengths in future sections. Also, this ds notation will be a nice notation for the next section as well.

Now that we've derived the arc length formula let's work some examples.

Example 1 Determine the length of $y = \ln(\sec x)$ between $0 \le x \le \frac{\pi}{4}$.

Show Solution

Example 2 Determine the length of $x = \frac{2}{3}(y-1)^{\frac{3}{2}}$ between $1 \le y \le 4$.

Show Solution

As noted in the last example we really do have a choice as to which ds we use. Provided we can get the function in the form required for a particular ds we can use it. However, as also noted above, there will often be a significant difference in difficulty in the resulting integrals. Let's take a quick look at what would happen in the previous example if we did put the function into the form $y = f(x)$.

Example 3 Redo the previous example using the function in the form $y = f(x)$ instead.

Show Solution

From a technical standpoint the integral in the previous example was not that difficult. It was just a Calculus I substitution. However, from a practical standpoint the integral was significantly more difficult than the integral we evaluated in Example 2. So, the moral of the story here is that we can use either formula (provided we can get the function in the correct form of course) however one will often be significantly easier to actually evaluate.

Okay, let's work one more example.

Example 4 Determine the length of $x = \frac{1}{2}y^2$ for $0 \le x \le \frac{1}{2}$. Assume that y is positive.

Show Solution

The first couple of examples ended up being fairly simple Calculus I substitutions. However, as this last example had shown we can end up with trig substitutions as well for these integrals.