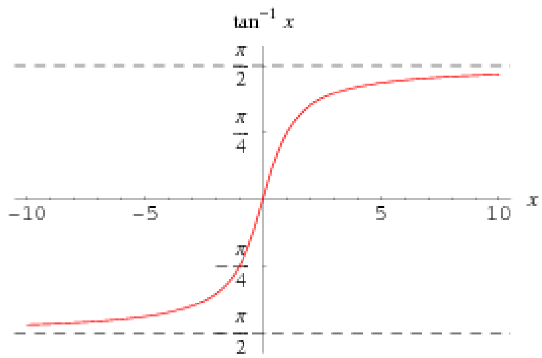
Inverse Tangent

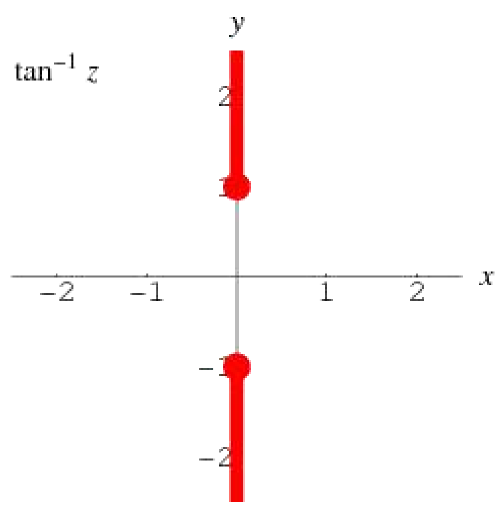
The inverse tangent is the multivalued function $\tan^{-1} z$ (Zwillinger 1995, p. 465), also denoted $\arctan z$ (Abramowitz and Stegun 1972, p. 79; Harris and Stocker 1998, p. 311; Jeffrey 2000, p. 124) or $\operatorname{arctg} z$ (Spanier and Oldham 1987, p. 333; Gradshteyn and Ryzhik 2000, p. 208; Jeffrey 2000, p. 127), that is the inverse function of the tangent. The variants $\operatorname{Arctan} z$ (e.g., Bronshtein and Semendyayev, 1997, p. 70) and $\operatorname{Tan}^{-1} z$ are sometimes used to refer to explicit principal values of the inverse cotangent, although this distinction is not always made (e.g,. Zwillinger 1995, p. 466).



The inverse tangent function $\tan^{-1} x$ is plotted above along the real axis.

Worse yet, the notation $\arctan z$ is sometimes used for the principal value, with $\operatorname{Arctan} z$ being used for the multivalued function (Abramowitz and Stegun 1972, p. 80). Note that in the notation $\tan^{-1} z$ (commonly used in North America and in pocket calculators worldwide), $\tan z$ denotes the tangent and -1 the inverse function, *not* the multiplicative inverse.

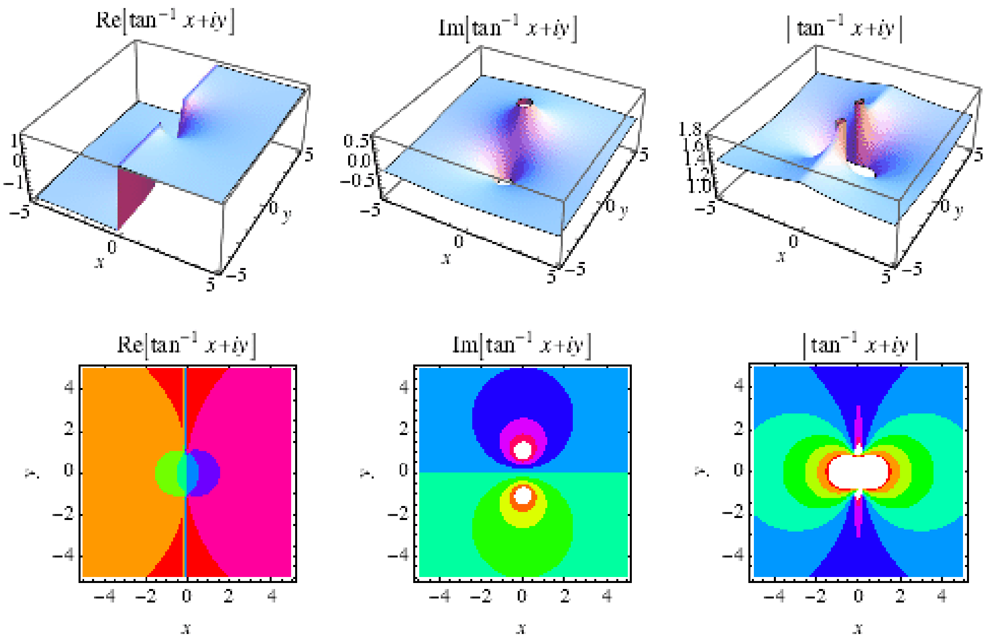
The principal value of the inverse tangent is implemented as `ArcTan`[*z*] in the Wolfram Language. In the GNU C library, it is implemented as `atan`(*double x*).



The inverse tangent is a multivalued function and hence requires a branch cut in the complex plane, which the Wolfram Language's convention places at $(-i\infty, -i]$ and $[i, i\infty)$. This follows from the definition of $\tan^{-1} z$ as

$$\tan^{-1} z = \frac{1}{2} i [\ln(1 - i z) - \ln(1 + i z)].$$

In the Wolfram Language (and in this work), this branch cut definition determines the range of $\tan^{-1} x$ for real x as $(-\pi/2, \pi/2)$. Care must be taken, however, as other branch cut definitions can give different ranges (most commonly, $(0, \pi)$).



	Min	Max	
Re	-5	5	
Im	-5	5	Replot

The inverse tangent function $\tan^{-1} z$ is plotted above in the complex plane.

$\tan^{-1} z$ has the special values

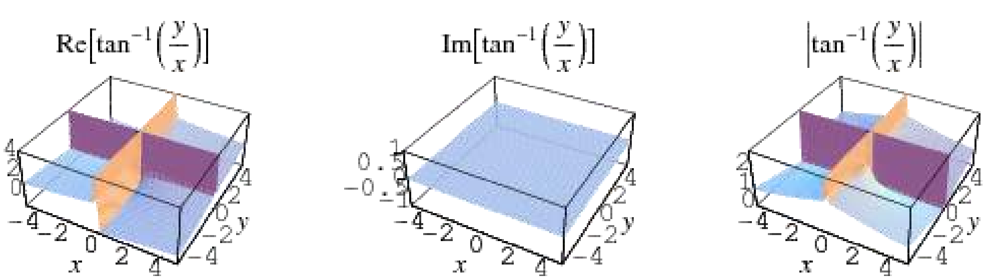
$$\begin{aligned} \tan^{-1}(-\infty) &= -\tfrac{1}{2}\pi \\ \tan^{-1}(-i) &= -i\,\infty \\ \tan^{-1} 0 &= 0 \\ \tan^{-1} i &= i\,\infty \\ \tan^{-1} \infty &= \tfrac{1}{2}\pi. \end{aligned}$$

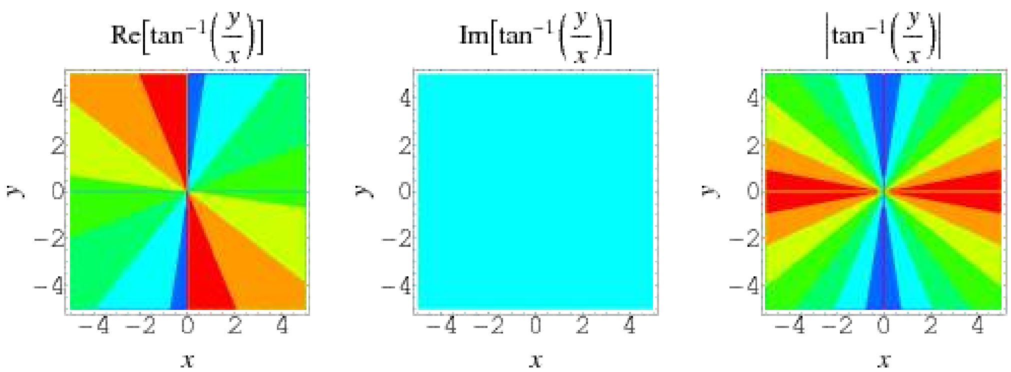
The derivative of $\tan^{-1} z$ is

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$$

and the indefinite integral is

$$\int \tan^{-1} z\,dz = z\tan^{-1} z - \tfrac{1}{2}\ln\left(1+z^2\right) + C.$$

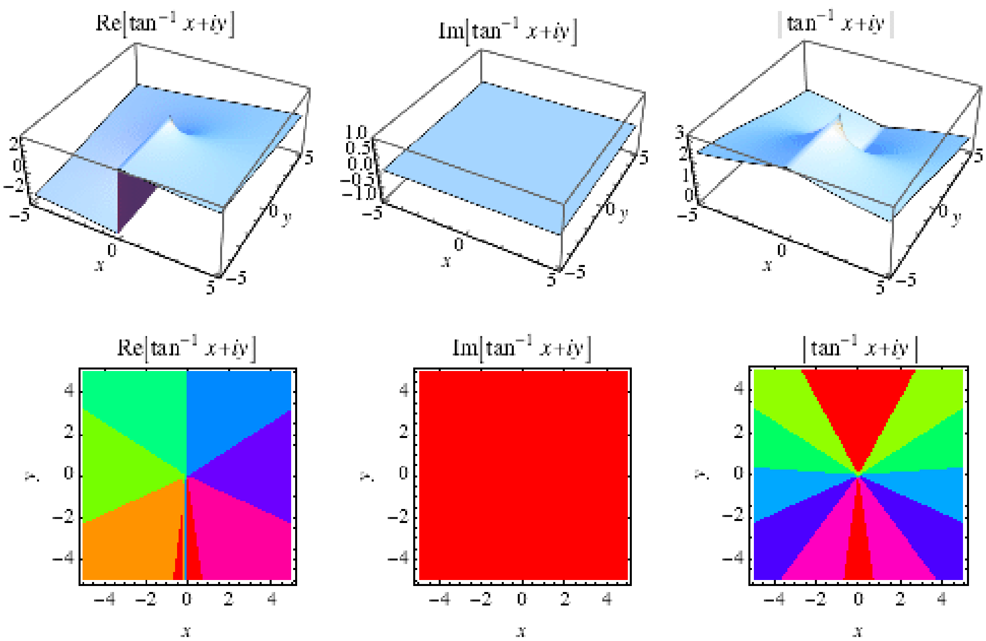




The complex argument of a complex number $z = x + i\,y$ is often written as

$$\theta = \tan^{-1}\left(\frac{y}{x}\right), \qquad (9$$

where θ, sometimes also denoted ϕ, corresponds to the counterclockwise angle from the positive real axis, i.e., the value of θ such that $x = \cos\theta$ and $y = \sin\theta$. Plots of $\tan^{-1}(y/x)$ are illustrated above for real values of x and y.



	Min	Max
Re	-5	5
Im	-5	5

Replot

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A special kind of inverse tangent that takes into account the quadrant in which z lies and is returned by the FORTRANcommand `ATAN2(y, x)`, the GNU C library command `atan2`(*double y*, *double x*), and the Wolfram Language command`ArcTan`[*x*, *y*], and is often restricted to the range $-\pi < \theta \leq \pi$. In the degenerate case when $x = 0$,

$$\phi = \begin{cases} -\frac{1}{2}\pi & \text{if } y < 0 \\ \text{undefined} & \text{if } y = 0 \\ \frac{1}{2}\pi & \text{if } y > 0. \end{cases} \qquad (10$$

The usual $\tan^{-1} z$ has the Maclaurin series of

$$\tan^{-1} z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1} \qquad (1$$

$$= z - \frac{1}{3} z^3 + \frac{1}{5} z^5 - \frac{1}{7} z^7 + \ldots \quad (1$$

(OEIS A033999 and A005408). A more rapidly converging form due to Euler is given by

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{2^{2n} (n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}} \quad (1$$

for real x (Castellanos 1988). This is related to the formula of Euler given by

$$\tan^{-1} x = \frac{y}{x} \left(1 + \frac{2}{3} y + \frac{2 \cdot 4}{3 \cdot 5} y^2 + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7} y^3 + \ldots\right), \quad (1$$

where

$$y \equiv \frac{x^2}{1+x^2}. \quad (1$$

The inverse tangent formulas are connected with many interesting approximations to pi

$$\tan^{-1}(1+x) = \frac{\pi}{4} + i \sum_{n=1}^{\infty} \frac{(-1-i)^n - (i-1)^n}{2^{n+1} n} x^n \quad (1$$

$$= \frac{1}{4} \pi + \frac{1}{2} x - \frac{1}{4} x^2 + \frac{1}{12} x^3 - \frac{1}{40} x^5 + \frac{1}{48} x^6 - \frac{1}{112} x^7 + \ldots \quad (1$$

(OEIS A075553 and A075554).

The inverse tangent satisfies

$$\tan^{-1} z = \cot^{-1} \left(\frac{1}{z}\right) \quad (1$$

for $z \neq 0$,

$$\tan^{-1} z = -\tan^{-1}(-z) \quad (1$$

for all complex z,

$$\tan^{-1} x = \frac{1}{2} \pi - \cos^{-1} \left(\frac{x}{\sqrt{x^2+1}}\right) \quad (2$$

$$= \sin^{-1} \left(\frac{x}{\sqrt{x^2+1}}\right) \quad (2$$

$$= \csc^{-1} \left(\frac{\sqrt{x^2+1}}{x}\right) \quad (2$$

for all real x, where equality for the last equation is understood to be in the limit as $x \to 0$, and

$$\tan^{-1} x = \begin{cases} -\frac{1}{2} \pi - \tan^{-1} \left(\frac{1}{x}\right) & \text{for } x < 0 \\ \frac{1}{2} \pi - \tan^{-1} \left(\frac{1}{x}\right) & \text{for } x > 0 \end{cases} \quad (2$$

$$= \begin{cases} -\frac{1}{2} \pi + \cot^{-1}(-x) & \text{for } x < 0 \\ \frac{1}{2} \pi + \cot^{-1}(-x) & \text{for } x > 0 \end{cases} \quad (2$$

$$= \begin{cases} -\frac{1}{2} \pi - \cot^{-1} x & \text{for } x < 0 \\ \frac{1}{2} \pi - \cot^{-1} x & \text{for } x > 0 \end{cases} \quad (2$$

$$= \quad (2$$

$$\begin{cases} -\cos^{-1}\left(\dfrac{1}{\sqrt{x^2+1}}\right) & \text{for } x<0 \\ \cos^{-1}\left(\dfrac{1}{\sqrt{x^2+1}}\right) & \text{for } x>0 \end{cases}$$

$$= \begin{cases} -\sec^{-1}\left(\sqrt{x^2+1}\right) & \text{for } x<0 \\ \sec^{-1}\left(\sqrt{x^2+1}\right) & \text{for } x>0. \end{cases}$$ (2

In terms of the hypergeometric function,

$$\tan^{-1} z = z\, {}_2F_1\left(1, \tfrac{1}{2}; \tfrac{3}{2}; -z^2\right)$$ (2

for complex z, and

$$\tan^{-1} x = \frac{x}{1+x^2}\, {}_2F_1\left(1, 1; \tfrac{3}{2}; \frac{x^2}{1+x^2}\right)$$ (2

for real x (Castellanos 1988).

Castellanos (1986, 1988) also gives some curious formulas in terms of the Fibonacci numbers,

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}\, t^{2n+1}}{5^n (2n+1)}$$ (3

$$= 5 \sum_{n=0}^{\infty} \frac{(-1)^n F_{2n+1}^2}{(2n+1)\left(u+\sqrt{u^2+1}\right)^{2n+1}}$$ (3

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+2} F_{2n+1}^3}{(2n+1)\left(v+\sqrt{v^2+5}\right)^{2n+1}},$$ (3

where

$$t \equiv \frac{2x}{1+\sqrt{1+\frac{4x^2}{5}}}$$ (3

$$u \equiv \frac{5}{4x}\left(1+\sqrt{1+\tfrac{24}{25}x^2}\right),$$ (3

and v is the largest positive root of

$$8xv^4 - 100v^3 - 450xv^2 + 875v + 625x = 0.$$ (3

The inverse tangent satisfies the addition formula

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$ (3

for $-1 < x, y < 1$, as well as the more complicated formula

$$\tan^{-1}\left(\frac{1}{a}\right) = 2\tan^{-1}\left(\frac{1}{2a}\right) - \tan^{-1}\left(\frac{1}{4a^3+3a}\right)$$ (3

valid for all complex a. An additional identity known to Euler is given by

(3

$$\tan^{-1}\left(\frac{1}{a-b}\right) = \tan^{-1}\left(\frac{1}{a}\right) + \tan^{-1}\left(\frac{b}{a^2 - a\,b + 1}\right)$$

for $(a > b \wedge a > 0)$ or $(a < b \wedge a < 0)$. Another interesting inverse tangent identity attributed to Charles Dodgson (Lewis Carroll) by Lehmer (1938b; Bromwich 1991, Castellanos 1988) is

$$\tan^{-1}(p+r) + \tan^{-1}(p+q) - \tan^{-1} p = \tfrac{1}{2}\pi, \quad (39)$$

where

$$1 + p^2 = q\,r \quad (40)$$

and $p, q, r > 0$.

The inverse tangent has continued fraction representations

$$\tan^{-1} x = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{4x^2}{5 + \cfrac{9x^2}{7 + \cfrac{16x^2}{9 + \ldots}}}}} \quad (41)$$

(Lambert 1770; Lagrange 1776; Wall 1948, p. 343; Olds 1963, p. 138) and

$$\tan^{-1} x = \cfrac{x}{1 + \cfrac{x^2}{3 - x^2 + \cfrac{9x^2}{5 - 3x^2 + \cfrac{25x^2}{7 - 5x^2 + \ldots}}}} \quad (42)$$

due to Euler and sometimes known as Euler's continued fraction (Borwein *et al.* 2004, p. 30).

To find $\tan^{-1} x$ numerically, the following arithmetic-geometric mean-like algorithm can be used. Let

$$a_0 = (1 + x^2)^{-1/2} \quad (43)$$
$$b_0 = 1. \quad (44)$$

Then compute

$$a_{i+1} = \tfrac{1}{2}(a_i + b_i) \quad (45)$$
$$b_{i+1} = \sqrt{a_{i+1}\,b_i}\,, \quad (46)$$

and the inverse tangent is given by

$$\tan^{-1} x = \lim_{n\to\infty} \frac{x}{\sqrt{1+x^2}\;a_n} \quad (47)$$

(Acton 1990).

An inverse tangent $\tan^{-1} n$ with integral n is called reducible if it is expressible as a finite sum of the form

$$\tan^{-1} n = \sum_{k=1} f_k \tan^{-1} n_k, \quad (48)$$

where f_k are positive or negative integers and n_i are integers $< n$. $\tan^{-1} m$ is reducible iff all the prime factors of $1 + m^2$ occur among the prime factors of $1 + n^2$ for $n = 1, \ldots, m-1$. A second necessary and sufficient condition is that the largest prime factor of $1 + m^2$ is less than $2\,m$. Equivalent to the second condition is the statement that every Gregory number $t_x = \cot^{-1} x$ can be uniquely expressed as a sum in terms of t_ms for which m is a Størmer number (Conway and Guy 1996). To find this decomposition, write

$$\arg(1 + i\,n) = \arg \prod_{k=1} (1 + n_k\,i)^{f_k}, \quad (49)$$

so the ratio

(50)

$$r = \frac{\prod_{k=1} (1 + n_k\, i)^{f_k}}{1 + i\, n}$$

is a rational number. Equation (50) can also be written

$$r^2 \left(1 + n^2\right) = \prod_{k=1} \left(1 + n_k^2\right)^{f_k}. \qquad (51$$

Writing (◇) in the form

$$\tan^{-1} n = \sum_{k=1} f_k \tan^{-1} n_k + f \tan^{-1} 1 \qquad (52$$

allows a direct conversion to a corresponding inverse cotangent formula

$$\cot^{-1} n = \sum_{k=1} f_k \cot^{-1} n_k + c \cot^{-1} 1, \qquad (53$$

where

$$c = 2 - f - 2 \sum_{k=1} f_r. \qquad (54$$

Todd (1949) gives a table of decompositions of $\tan^{-1} n$ for $n \leq 342$. Conway and Guy (1996) give a similar table in terms of Størmer numbers.

Arndt and Gosper give the remarkable inverse tangent identity

$$\sin\left(\sum_{k=1}^{2n+1} \tan^{-1} a_k\right) = \frac{(-1)^n}{2n+1} \frac{\sum_{k=1}^{2n+1} \prod_{j=1}^{2n+1} \left[a_j - \tan\left(\frac{\pi(j-k)}{2n+1}\right)\right]}{\sqrt{\prod_{j=1}^{2n+1} \left(a_j^2 + 1\right)}}. \qquad (55$$

There is an amazing set of BBP-type formulas for $\tan^{-1}(4/5)$:

$$\begin{aligned}
\tan^{-1}\left(\tfrac{4}{5}\right) = {} & \frac{1}{131072} \sum_{k=0}^{\infty} \frac{1}{1048576^k} \left[\frac{262144}{40k+2} - \frac{163840}{40k+5} - \frac{65536}{40k+6} + \right. \\
& \frac{16384}{40k+10} - \frac{4096}{40k+14} - \frac{5120}{40k+15} + \frac{1024}{40k+18} - \frac{256}{40k+22} + \\
& \left. \frac{160}{40k+25} + \frac{64}{40k+26} - \frac{16}{40k+30} + \frac{4}{40k+34} + \frac{5}{40k+35} - \frac{1}{40k+38}\right] \qquad (5 \\
= {} & \frac{1}{131072} \sum_{k=0}^{\infty} \frac{1}{1048576^k} \left[\frac{393216}{40k+4} + \frac{163840}{40k+5} - \frac{131072}{40k+6} - \frac{163840}{40k+8} + \frac{24576}{40k+12} - \right. \\
& \frac{8192}{40k+14} - \frac{15360}{40k+15} - \frac{10240}{40k+16} - \frac{1024}{40k+20} - \frac{512}{40k+22} - \frac{640}{40k+24} - \\
& \left. \frac{160}{40k+25} + \frac{96}{40k+28} - \frac{32}{40k+30} - \frac{40}{40k+32} + \frac{15}{40k+35} + \frac{6}{40k+36} - \frac{2}{40k+38}\right] \qquad (5 \\
= {} & \frac{1}{131072} \sum_{k=0}^{\infty} \frac{1}{1048576^k} \left[\frac{262144}{40k+1} - \frac{262144}{40k+3} - \frac{65536}{40k+5} - \frac{327680}{40k+6} + \frac{65536}{40k+7} - \right. \\
& \frac{163840}{40k+8} + \frac{16384}{40k+9} - \frac{40960}{40k+10} - \frac{16384}{40k+11} - \frac{4096}{40k+13} - \frac{20480}{40k+14} - \frac{16384}{40k+15} - \\
& \frac{10240}{40k+16} + \frac{1024}{40k+17} - \frac{1024}{40k+19} - \frac{2560}{40k+20} - \frac{256}{40k+21} - \frac{1280}{40k+22} + \\
& \frac{256}{40k+23} - \frac{640}{40k+24} + \frac{64}{40k+25} - \frac{64}{40k+27} - \frac{16}{40k+29} - \frac{40}{40k+30} + \\
& \left. \frac{16}{40k+31} - \frac{40}{40k+32} + \frac{4}{40k+33} + \frac{16}{40k+35} - \frac{1}{40k+37} - \frac{5}{40k+38} + \frac{1}{40k+39}\right] \qquad (5 \\
= {} &
\end{aligned}$$

$$\frac{1}{262\,144}\sum_{k=0}^{\infty}\frac{1}{1\,048\,576^{k}}\left[\frac{262\,144}{40\,k+3}+\frac{262\,144}{40\,k+4}+\frac{131\,072}{40\,k+6}-\frac{65\,536}{40\,k+7}+\frac{81\,920}{40\,k+10}+\frac{16\,384}{40\,k+11}+\frac{16\,384}{40\,k+12}+\frac{8192}{40\,k+14}-\frac{4096}{40\,k+15}+\frac{1024}{40\,k+19}+\frac{1024}{40\,k+20}+\frac{512}{40\,k+22}-\frac{256}{40\,k+23}+\frac{64}{40\,k+27}+\frac{64}{40\,k+28}-\frac{48}{40\,k+30}-\frac{16}{40\,k+31}+\frac{4}{40\,k+35}+\frac{4}{40\,k+36}+\frac{2}{40\,k+38}-\frac{1}{40\,k+39}\right],$$