Formulas and Theorems for Reference

I. Trigonometric Formulas

1. $\sin^2\theta + \cos^2\theta = 1$

2. $1 + \tan^2\theta = \sec^2\theta$

3. $1 + \cot^2\theta = \csc^2\theta$

4. $\sin(-\theta) = -\sin\theta$

5. $\cos(-\theta) = \cos\theta$

6. $\tan(-\theta) = -\tan\theta$

7. $\sin(A + B) = \sin A\cos B + \sin B\cos A$

8. $\sin(A - B) = \sin A\cos B - \sin B\cos A$

9. $\cos(A + B) = \cos A\cos B - \sin A\sin B$

10. $\cos(A - B) = \cos A\cos B + \sin A\sin B$

11. $\sin 2\theta = 2\sin\theta\cos\theta$

12. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

13. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\cot\theta}$

14. $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$

15. $\sec\theta = \dfrac{1}{\cos\theta}$

16. $\csc\theta = \dfrac{1}{\sin\theta}$

17. $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$

18. $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

II. Differentiation Formulas

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(fg) = fg' + gf'$

3. $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$

4. $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

5. $\frac{d}{dx}(\sin x) = \cos x$

6. $\frac{d}{dx}(\cos x) = -\sin x$

7. $\frac{d}{dx}(\tan x) = \sec^2 x$

8. $\frac{d}{dx}(\cot x) = -\csc^2 x$

9. $\frac{d}{dx}(\sec x) = \sec x \tan x$

10. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

11. $\frac{d}{dx}(e^x) = e^x$

12. $\frac{d}{dx}(a^x) = a^x \ln a$

13. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

14. $\frac{d}{dx}(\text{Arcsin } x) = \frac{1}{\sqrt{1 - x^2}}$

15. $\frac{d}{dx}(\text{Arcsin } x) = \frac{1}{1 + x^2}$

III. Integration Formulas

1. $\int a\ dx = ax + C$

2. $\int \frac{1}{x}\ dx = \ln|x| + C$

3. $\int e^x\ dx = e^x + C$

4. $\int a^x\ dx = \frac{a^x}{\ln a} + C$

5. $\int \ln x\ dx = x \ln x - x + C$

6. $\int \sin x\ dx = -\cos x + C$

7. $\int \cos x\ dx = \sin x + C$

8. $\int \tan x\ dx = \ln|\sec x| + C \text{ or } -\ln|\cos x| + C$

9. $\int \cot x\ dx = \ln|\sin x| + C$

10. $\int \sec x\ dx = \ln|\sec x + \tan x| + C$

11. $\int \csc x\ dx = \ln|\csc x - \cot x| + C$

12. $\int \sec^2 x\ dx = \tan x + C$

13. $\int \sec x \tan x\ dx = \sec x + C$

14. $\int \csc^2 x\ dx = -\cot x + C$

15. $\int \csc x \cot x\ dx = -\csc x + C$

16. $\int \tan^2 x\ dx = \tan x - x + C$

17. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \text{Arctan}\left(\frac{x}{a}\right) + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \text{Arcsin}\left(\frac{x}{a}\right) + C$

IV. Formulas and Theorems

1. Limits and Continuity

A function $y = f(x)$ is continuous at $x = a$ if:

i) $f(a)$ is defined (exists)

ii) $\lim_{x \to a} f(x)$ exists, and

iii) $\lim_{x \to a} f(x) = f(a)$

Otherwise, f is discontinuous at $x = a$.

The limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal — that is,

$$\lim_{x \to a} f(x) = L \Longleftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

2. Intermediate - Value Theroem

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.

Note: If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a, b).

3. Limits of Rational Functions as $x \to \pm\infty$

1. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^2 - 2x}{x^3 + 3} = 0$

2. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite if the degree of $f(x) >$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^3 + 2x}{x^2 - 8} = \infty$

3. $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

Note: The limit will be the ratio of the leading coefficient of $f(x)$ to $g(x)$.

Example: $\lim_{x \to \infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$

4. Horizontal and Vertical Asymptotes

   1. A line $y = b$ is a horizontal asymptote of the graph of $y = f(x)$ if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.

   2. A line $x = a$ is a vertical asymptote of the graph of $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

5. Average and Instantaneous Rate of Change

   1. Average Rate of Change: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y}{\Delta x}.$$

   2. Instantaneous Rate of Change: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

6. Definition of the Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The latter definition of the derivative is the instantaneous rate of change of $f(x)$ with respect to x at $x = a$.

Geometrically, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point.

7. The Number e as a limit

   1. $\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e$

   2. $\lim_{n \to 0} (1 + n)^{\frac{1}{n}} = e$

8. Rolle's Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is at least one number c in the open interval (a, b) such that $f'(c) = 0$.

9. Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one number c in (a, b) such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

10. Extreme - Value Theorem

If f is continuous on a closed interval $[a, b]$, then $f(x)$ has both a maximum and a minimum on $[a, b]$.

11. To find the maximum and minimum values of a function $y = f(x)$, locate

1. the point(s) where $f'(x)$ changes sign. To find the candidates first find where $f'(x) = 0$ or is infinite or does not exist.

2. the end points, if any, on the domain of $f(x)$.

Compare the function values at all of these points to find the maximums and minimums.

12. Let f be differentiable for $a < x < b$ and continuous for $a \le x \le b$.

1. If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.

2. If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.

13. Suppose that $f''(x)$ exists on the interval (a, b).

1. If $f''(x) > 0$ in (a, b), then f is concave upward in (a, b).

2. If $f''(x) < 0$ in (a, b), then f is concave downward in (a, b).

To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ or where $f''(x)$ fails to exist. These are the only cndidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

14. Differentiability implies continuity: If a function is differentiable at a point $x = a$, it is continuous at that point. The converse is false, i.e. continuity does not imply differentiability.

15. Local Linearity and Linear Approximation

The linear approximation of $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$.

To estimate the slope of a graph at a point draw a tangent line to the graph at that point. Another way is (by using a graphics calculator) to "zoom in" around the point in question until the graph "looks" straight. This method almost always works. If we "zoom in" and the graph looks straight at a point, say $x = a$, then the function is locally linear at that point.

The graph of $y = |x|$ has a sharp corner at $x = 0$. This corner cannot be smoothed out be "zooming in" repeatedly. Consequently, the derivative of $|x$ does not exist at $x = 0$, hence, is not locally linear at $x = 0$.

16. Comparing Rates of Change

The exponential function $y = e^x$ grows very rapidly as $x \to \infty$ while the logarithmic function $y = \ln x$ grows very slowly as $x \to \infty$.

Exponential functions like $y = 2^x$ or $y = e^x$ grow more rapidly as $x \to \infty$ than any positive power of x. The function $y = \ln x$ grows slower as $x \to \infty$ than any nonconstant polynomial.

We say, that as $x \to \infty$:

1. $f(x)$ grows faster than $g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ or if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0$.
   If $f(x)$ grows faster than $g(x)$ as $x \to \infty$, then $g(x)$ grows slower than $f(x)$ as $x \to \infty$.

2. $f(x)$ and $g(x)$ grow at the same rate as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \neq 0$ (L is finite and nonzero).

For example,

1. e^x grows faster than x^3 as $x \to \infty$ since $\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$

2. x^4 grows faster than $\ln x$ as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^4}{\ln x} = \infty$

3. $x^2 + 2x$ grows at the same rate as x^2 as $x \to \infty$ since $\lim_{x\to\infty} \frac{x^2 + 2x}{x^2} = 1$

To find some of these limits as $x \to \infty$, you may use the graphing calculator. Make sure that an appropriate viewing window is used.

17. Inverse Functions

1. If f and g are two functions such that $f(g(x)) = x$ for every x in the domain of g, and, $g(f(x)) = x$, for every x in the domain of f, then, f and g are inverse functions of each other.

2. A function f has an inverse function if and only if no horizontal line intersects its graph more than once.

3. If f is either increasing or decreasing in an interval, then f has an inverse function over that interval.

4. If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(\text{I})$ and $g'(f(x)) = \frac{1}{f'(x)}$.

18. Properties of e^x

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers, $-\infty < x < \infty$.
3. The range is the set of all positive numbers, $y > 0$.
4. $\frac{d}{dx}(e^x) = e^x$.
5. $y = e^x$ is continuous, increasing, and concave up for all x.
6. $\lim_{x \to +\infty} e^x = +\infty$ and $\lim_{x \to -\infty} e^x = 0$.
7. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

19. Properties of $\ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.
2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.
3. $y = \ln x$ is continuous, increasing, and concave down everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$.
5. $\ln(a/b) = \ln a - \ln b$
6. $\ln a^r = r \ln a$
7. $y = \ln x < 0$ if $0 < x < 1$ and $\ln x > 0$ if $x > 1$.
8. $\lim_{x \to +\infty} \ln x = +\infty$ and $\lim_{x \to 0^+} \ln x = -\infty$.
9. $\log_a x = \frac{\ln x}{\ln a}$

20. Trapezoidal Rule

If a function f is continuous on the closed interval $[a, b]$ where $[a, b]$ has been partitioned into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, each of length $(b-a)/n$, then $\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$.

The Trapezoidal Rule is the average of the left-hand and right-hand Riemann sums.

21. Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

1. $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$, c is a nonzero constant.

2. $\int_a^a f(x)\,dx = 0$

3. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

4. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c, regardless of the order a, b, and c.

5. If $f(x)$ is an odd function, then $\int_{-a}^a f(x)\,dx = 0$

6. If $f(x)$ is an even function, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

7. If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$

8. If $g(x) \geq f(x)$ on $[a,b]$, then $\int_a^b g(x)\,dx \geq \int_a^b f(x)\,dx$

22. Definition of Definite Integral as the Limit of a Sum

Suppose that a function $f(x)$ is continuous on the closed interval $[a,b]$. Divide the interval into n equal subintervals, of length $\Delta x = \frac{b-a}{n}$. Choose one number in each subinterval i.e., x_1 in the first, x_2 in the second, $\ldots$, x_k in the kth, $\ldots$, and x_n in the nth. Then $\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\,dx$.

23. Fundamental Theorem of Calculus

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$

or $\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$ and $\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x)$.

24. Velocity, Speed, and Acceleration

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change.

2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.

   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.

3. The acceleration is the instantaneous rate of change of velocity – it is the derivative of the velocity – that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing. The acceleration gives the rate at which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity $= v(t) = x'(t) = \dfrac{dx}{dt}$

ii) acceleration $= a(t) = x''(t) = v'(t) = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$

iii) $v(t) = \displaystyle\int a(t)\, dt$

iv) $x(t) = \displaystyle\int v(t)\, dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is Average Velocity $= \dfrac{\text{Change in position}}{\text{Length of time}} = \dfrac{s(t) - s(t_0)}{t - t_0}$, where $s(t)$ is the position of the particle at time t.

25. The average value of $f(x)$ on $[a, b]$ is $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\, dx$.

26. Area Between Curves

If f and g are continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, then the area between the curves is $\displaystyle\int_a^b [f(x) - g(x)]\, dx$.

27. Volume of Solids of Revolution

Let f be nonnegative and continuous on $[a, b]$, and let R be the region bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$.

When this region R is revolved about the x-axis, it generates a solid (having circular cross sections) whose volume $V = \int_a^b \pi[f(x)]^2 \, dx$.

28. Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$, taken perpendicular to the x-axis, volume $= \int_a^b A(x) \, dx$.

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis, volume $= \int_c^d A(y) \, dy$.

29. Solving Differential Equations: Graphically and Numerically

Slope Fields

At every point (x, y) a differential equation of the form $\frac{dy}{dx} = f(x, y)$ gives the slope of the member of the family of solutions that contains that point. A slope field is a graphical representation of this family of curves. At each point in the plane, a short segment is drawn whose slope is equal to the value of the derivative at that point. These segments are tangent to the solution's graph at the point.

The slope field allows you to sketch the graph of the solution curve even though you do not have its equation. This is done by starting at any point (usually the point given by the initial condition), and moving fron one point to the next in the direction indicated by the segments of the slope field.

Some calculators have built in operations for drawing slope fields; for calculators without this feature there are programs available for drawing them.

30. Solving Differential Equations by Separating the Variables

There are many techniques for solving differential equations. Any differential equation you may be asked to solve on the AB Calculus Exam can be solved by separating the variables. Rewrite the equation as an equivalent equation with all the x and dx terms on one side and all the y and dy terms on the other. Antidifferentiate both sides to obtain an equation without dx or dy, but with one constant of integration. Use the initial condition to evaluate this constant.