Cosecant, Secant & Cotangent

In this unit we explain what is meant by the three trigonometric ratios cosecant, secant and cotangent. We see how they can appear in trigonometric identities and in the solution of trigonometrical equations. Finally, we obtain graphs of the functions cosec θ, $\sec\theta$ and $\cot\theta$ from knowledge of the related functions $\sin\theta$, $\cos\theta$ and $\tan\theta$.

In order to master the techniques explained here it is vital that you undertake the practice exercises provided.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- define the ratios cosecant, secant and cotangent
- plot graphs of cosec θ, $\sec\theta$ and $\cot\theta$

Contents

1. Introduction

This unit looks at three new trigonometric functions cosecant (cosec), secant (sec) and cotangent (cot). These are not entirely new because they are derived from the three functions sine, cosine and tangent.

2. Definitions of cosecant, secant and cotangent

These functions are defined as follows:

Key Point

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

These functions are useful in the solution of trigonometrical equations, they can appear in trigonometric identities, and they can arise in calculus problems, particularly in integration.

Example

Consider the trigonometric identity

$$\sin^2\theta + \cos^2\theta = 1$$

Suppose we divide everything on both sides by $\cos^2\theta$. Doing this produces

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

This can be rewritten as

$$\left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2$$

that is as

$$\tan^2\theta + 1 = \sec^2\theta$$

This, in case you are not already aware, is a common trigonometrical identity involving $\sec\theta$.

Example

Consider again the trigonometric identity

$$\sin^2\theta + \cos^2\theta = 1$$

Suppose this time we divide everything on both sides by $\sin^2\theta$; this produces

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

This can be rewritten as

$$1+\left(\frac{\cos\theta}{\sin\theta}\right)^2=\left(\frac{1}{\sin\theta}\right)^2$$

that is as

$$1+\cot^2\theta=\mathrm{cosec}^2\theta$$

Again, we see one of our new trigonometric functions, $\mathrm{cosec}\,\theta$, appearing in an identity.

Example

Suppose we wish to solve the trigonometrical equation

$$\cot^2\theta=3 \qquad \text{for } \; 0^\circ\leq\theta<360^\circ$$

We begin the solution by taking the square root:

$$\cot\theta=\sqrt{3} \qquad \text{or} \qquad -\sqrt{3}$$

It then follows that

$$\frac{1}{\tan\theta}=\sqrt{3} \qquad \text{or} \qquad -\sqrt{3}$$

Inverting we find

$$\tan\theta=\frac{1}{\sqrt{3}} \qquad \text{or} \qquad -\frac{1}{\sqrt{3}}$$

The angle whose tangent is $\dfrac{1}{\sqrt{3}}$ is one of the special angles described in the unit *Trigonometrical ratios in a right-angled triangle*. In fact $\dfrac{1}{\sqrt{3}}$ is the tangent of 30°. So this is one solution of the equation $\tan\theta=\frac{1}{\sqrt{3}}$. What about other solutions ?

We refer to a graph of the function $\tan\theta$ as shown in Figure 1.

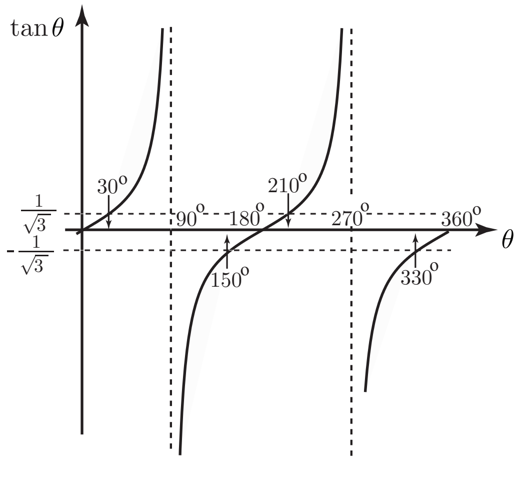

Figure 1. A graph of $\tan\theta$.

From the graph we see that the next solution of $\tan\theta=\frac{1}{\sqrt{3}}$ is 210° (that is 180° further along).

From the same graph we can also deduce, by consideration of symmetry, that the angles whose tangent is $-\dfrac{1}{\sqrt{3}}$ are 150° and 330°.

In summary, the equation $\cot^2\theta = 3$ has solutions

$$\theta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$$

So, solving equations involving cosec, sec and cot can often be solved by simply turning them into equations involving the more familiar functions sin, cos and tan.

3. The graph of cosec θ

We study the graph of cosec θ by first studying the graph of the closely related function $\sin\theta$, one cycle of the graph of which is shown in Figure 2.

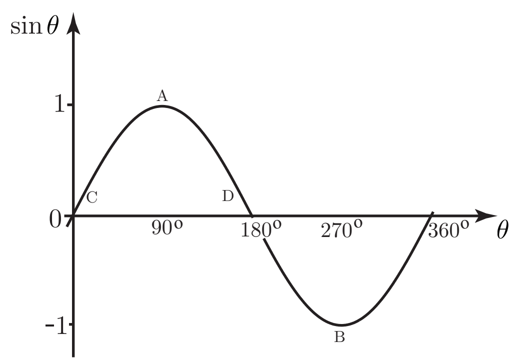

Figure 2. A graph of $\sin\theta$.

The graph of cosec θ can be deduced from the graph of $\sin\theta$ because cosec $\theta = \frac{1}{\sin\theta}$. Note that when $\theta = 90^\circ$, $\sin\theta = 1$ and hence cosec $\theta = 1$ as well. Similarly when $\theta = 270^\circ$, $\sin\theta = -1$ and hence cosec $\theta = -1$ as well. These observations enable us to plot two points on the graph of cosec θ. The corresponding points are marked A and B in both Figures 2 and 3. When $\theta = 0$, $\sin\theta = 0$, but because we can never divide by 0 we cannot evaluate cosec θ in this way. However, note that if θ is very small and positive (i.e. close to, but not equal to zero) $\sin\theta$ will be small and positive, and hence $\dfrac{1}{\sin\theta}$ will be large and positive. Points marked C on the graphs represent this. Similarly when $\theta = 180^\circ$, $\sin\theta = 0$ and again we cannot divide by zero to find cosec 180°. Suppose we look at values of θ just below 180°. Here, $\sin\theta$ is small and positive, so once again cosec θ will be large and positive (points D).

These observations enable us to gradually build up the graph as shown in Figure 3. The vertical dotted lines on the graph are called asymptotes.

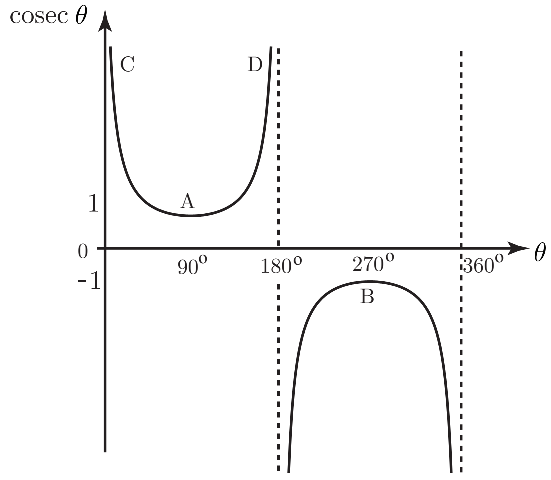

Figure 3. A graph of cosec θ.

Note that when θ is just slightly greater than $180°$ then $\sin\theta$ is small and negative, so that cosec θ is large and negative as shown in Figure 3. Continuing in this way the full graph of cosec θ can be constructed.

In Figure 2 we showed just one cycle of the sine graph. This generated one cycle of the graph of $\text{cosec}\,\theta$. Clearly, if further cycles of the sine graph are drawn these will generate further cycles of the cosecant graph. We conclude that the graph of $\text{cosec}\,\theta$ is periodic with period 2π.

4. The graph of sec θ

We can draw the graph of sec θ by first studying the graph of the related function $\cos\theta$ one cycle of which is shown in Figure 4.

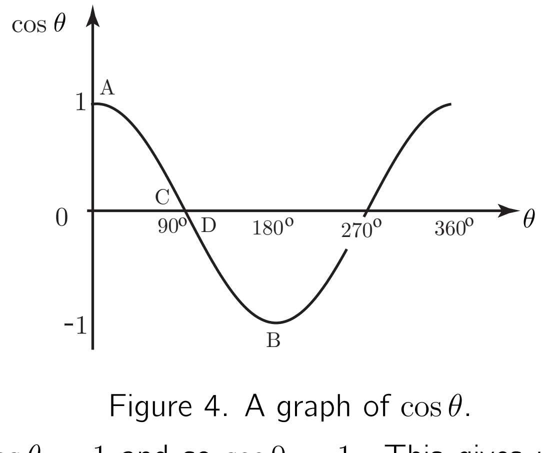

Figure 4. A graph of $\cos\theta$.

Note that when $\theta = 0$, $\cos\theta = 1$ and so $\sec 0 = 1$. This gives us a point (A) on the graph. Similarly when $\theta = 180°$, $\cos\theta = -1$ and so $\sec 180° = -1$ (Point B). When $\theta = 90°$, $\cos\theta = 0$ and so we cannot evaluate $\sec 90°$. We proceed as before and look a little to the left and right. When $\cos\theta$ is small and positive, $\frac{1}{\cos\theta}$ will be large and positive. This gives point C. When $\cos\theta$ is small and negative, $\frac{1}{\cos\theta}$ will be large and negative. This gives point D. Continuing in this way we can produce the graph shown in Figure 5.

Recall that we have only shown one cycle of the cosine graph in Figure 4. However because this repeats with a period of 2π it follows that the graph of $\sec\theta$ is also periodic with period 2π.

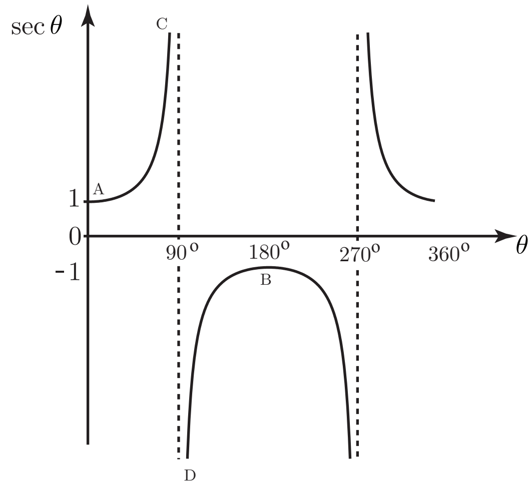

Figure 5. A graph of $\sec\theta$.

5. The graph of cot θ

We can draw the graph of cot θ by first studying the graph of $\tan\theta$ two cycles of which are shown in Figure 6.

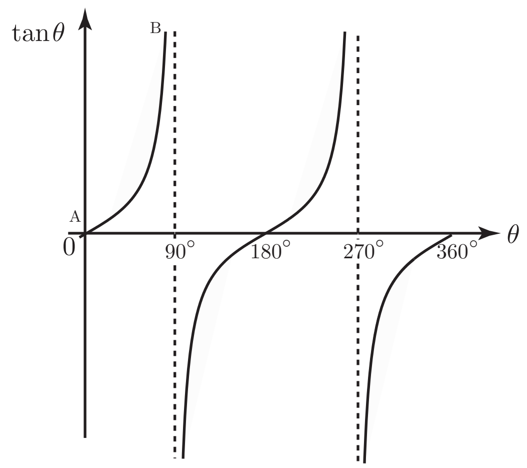

Figure 6. A graph of $\tan\theta$.

We proceed as before. When θ is small and positive (just above zero), so too is $\tan\theta$. So cot θ will be large and positive (point A). When θ is close to $90°$ the value of $\tan\theta$ is very large and positive, and so cot θ will be very small (point B). In this way we can obtain the graph shown in

Figure 7. Because the tangent graph is periodic with period π, so too is the graph of $\cot\theta$.

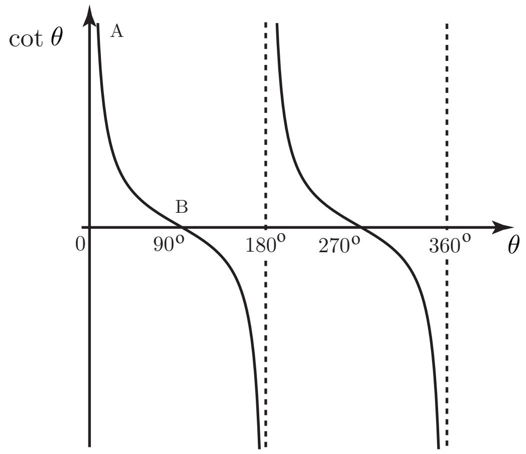

Figure 7. A graph of $\cot\theta$.

In summary, we have now met the three new trigonometric functions cosec, sec and cot and obtained their graphs from knowledge of the related functions sin, cos and tan.

Exercises

1. Use the values of the trigonometric rations of the special angles 30°, 45° and 60° to determine the following without using a calculator

a)	$\cot 45^o$	b)	$\text{cosec}\, 30^o$	c)	$\sec 60^o$
d)	$\text{cosec}^2 45^o$	e)	$\cot^2 60^o$	f)	$\sec^2 30^o$
g)	$\cot 315^o$	h)	$\text{cosec}\,(-30^o)$	i)	$\sec 240^o$

2. Find all the solutions of each of the following equations in the range stated (give your answers to 1 decimal place)

   (a) $\cot\theta = 0.2$ with $0^o < \theta < 360^o$

   (b) $\text{cosec}\,\theta = 4$ with $0^o < \theta < 180^o$

   (c) $\text{cosec}\,\theta = 4$ with $0^o < \theta < 360^o$

   (d) $\text{cosec}\,\theta = 4$ with $-180^o < \theta < 180^o$

   (e) $\sec\theta = 4$ with $0^o < \theta < 180^o$

   (f) $\sec\theta = 4$ with $0^o < \theta < 360^o$

   (g) $\sec\theta = 4$ with $-180^o < \theta < 180^o$

   (h) $\cot\theta = 0.5$ with $0^o < \theta < 360^o$

   (i) $\text{cosec}\,\theta = 0.5$ with $0^o < \theta < 360^o$

   (j) $\sec\theta = 0.5$ with $0^o < \theta < 360^o$

3. Determine whether each of the following statements is true or false

   (a) $\cot\theta$ is periodic with period 180°.

   (b) $\text{cosec}\,\theta$ is periodic with period 180°.

(c) Since the graph of $\cos\theta$ is continuous, the graph of $\sec\theta$ is continuous.

(d) $\operatorname{cosec}\theta$ never takes a value less than 1 in magnitude.

(e) $\cot\theta$ takes all values,

Answers

1. a) 1 b) 2 c) 2 d) 2 e) $\frac{1}{3}$ f) $\frac{4}{3}$ g) -1 h) -2 i) -2

2. a) 78.7^o, 258.7^o b) 14.5 ^{o}, 165.5 ^{o} c) 14.5 ^{o}, 165.5 ^{o} d) 14.5 ^{o}, 165.5 ^{o} e) 75.5^o f) 75.5^o, 284.5^o g) 75.5^o, -75.5^o h) 63.4^o, 243.4 ^{o} i) No solutions j) No solutions

3. a) True b) False c) False d) True e) True